MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2019

Calculator-assumed

Marking Key

The release date for this exam and marking scheme is 14th June.

Question 9(a) **(3 marks)**

Solution	
If $z^2+3z+9=0 \Rightarrow z=\frac{1}{2}(-3\pm\sqrt{9-36})=\frac{1}{2}(-3\pm 3\sqrt{3}i)$ Now $-3+3\sqrt{3}i$ has modulus $\sqrt{9+27}=6$ and argument $\pi-\tan^{-1}(\sqrt{3})=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$ Thus the two roots of the quadratic are $z=3\,\text{cis}\,\theta$ with $\theta=\pm\frac{2\pi}{3}$	
Mathematical behaviours	Marks
• applies the quadratic formula appropriately • identifies the modulus and argument for one of the roots • deduces the second root	1 1 1

Question 9(b) **(3 marks)**

Solution	
Since $z=3\exp\left(\pm\frac{2\pi i}{3}\right)$ then $z^N=3^N\exp\left(\pm\frac{2N\pi i}{3}\right)$ If $z_1^N=z_2^N$ then $\exp\left(\frac{2N\pi i}{3}\right)=\exp\left(-\frac{2N\pi i}{3}\right)\Rightarrow\exp\left(\frac{4N\pi i}{3}\right)=1$ For this to hold, the argument must be a multiple of 2π which implies that N is a multiple of 3	
Mathematical behaviours	Marks
• states the correct form for z^N • derives the correct equation for the two expressions to be equal • deduces the acceptable values of N	1 1 1

Question 10(a) **(3 marks)**

Solution	
If $\mathbf{r}(t) = 3\cos(2t)\,\mathbf{i} + 4\sin(2t)\,\mathbf{j}$ then $x = 3\cos 2t$ and $y = 4\sin 2t$ Hence $\cos 2t = x/3$ and $\sin 2t = y/4$. Since $\sin^2\theta + \cos^2\theta = 1$, $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1$	
Mathematical behaviours	**Marks**
• correctly determines equations for $\frac{x}{3} = \cos(2t)$ and $\frac{y}{4} = \sin(2t)$	2
• uses Pythagorean theorem correctly to obtain the result required	1

Question 10(b) **(1 mark)**

Solution	
The path is an ellipse	
Mathematical behaviours	**Marks**
• states the path is an ellipse	1

Question 10(c) **(1 mark)**

Solution	
If $\mathbf{r}(t) = 3\cos(2t)\,\mathbf{i} + 4\sin(2t)\,\mathbf{j}$ then $\mathbf{v}(t) = -6\sin(2t)\,\mathbf{i} + 8\cos(2t)\,\mathbf{j}$	
Mathematical behaviours	**Marks**
• derives the correct expression for the velocity vector	1

Question 10(d) **(3 marks)**

Solution	
Speed =|**v**(t)| This equals $\sqrt{(-6\sin 2t)^2 + (8\cos 2t)^2} = \sqrt{36\sin^2 2t + 64\cos^2 2t}$ $= \sqrt{36(1-\cos^2 2t) + 64\cos^2 2t}$ $= \sqrt{36 + 28\cos^2 2t}$	
	Marks
• writes down the correct expression for the speed • rewrites $\sin^2 2t = 1 - \cos^2 2t$ • obtains an expression for the speed in terms of $\cos^2 2t$	1 1 1

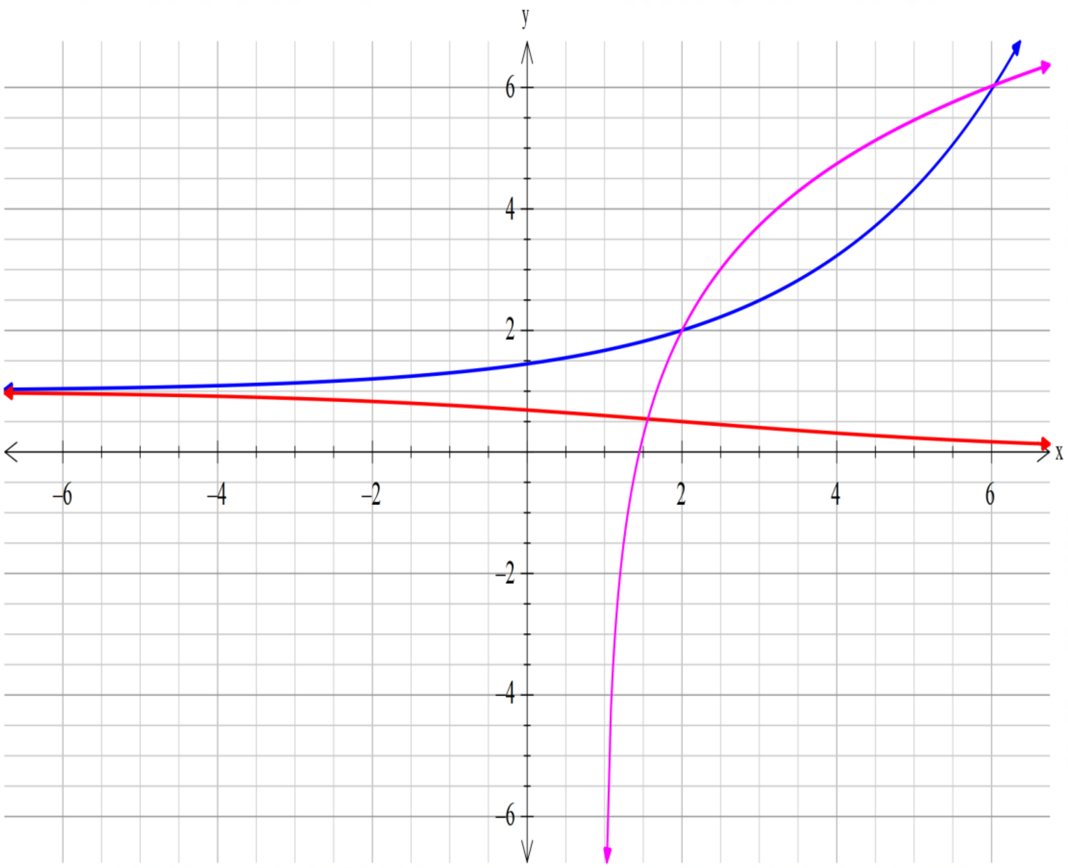
Question 10(e) **(2 marks)**

Solution
Maximum speed occurs when $\cos^2 2t = 1 \Rightarrow \cos 2t = \pm 1$ and then speed is $\sqrt{64} = 8$ If $\cos 2t = \pm 1 \quad \Rightarrow \quad 2t = n\pi \quad \Rightarrow \quad t = n\pi / 2$ for integer n Hence maximum speed attained when $t = n\pi / 2$ for integer values of n

	Marks
• deduces the maximum speed	1
• states the times when maximum speed is attained	1

Question 11 **(6 marks)**

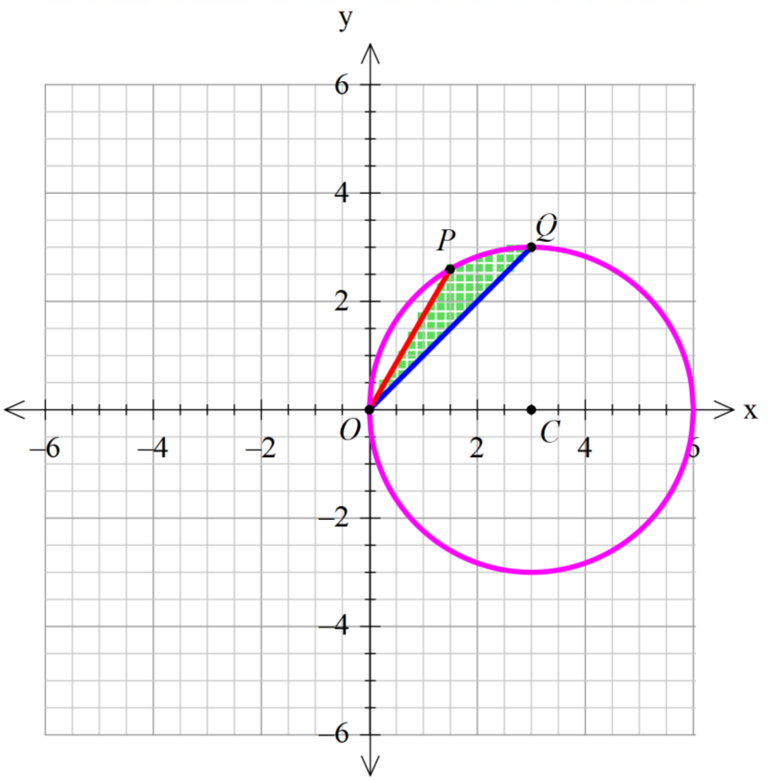
Solution



Mathematical behaviours	Marks
• displays correct shape of $y = 1/f(x)$ graph	1
• indicates correct horizontal 'limits' of $1/f(x)$ graph	1+1
• displays 'reflection' property of $y = f^{-1}(x)$ graph in $y = x$	1
• displays 'intersection' points near $(2,2)$ and $(6,6)$	1+1

Question 12 **(8 marks)**

Solution



As OC=CQ and angles $COQ = CQO = \pi/4$; hence $OCQ = \pi/2$ so co-ordinates of Q are (3,3)

Also angles $POC = OPC$ and since $CP = CO$ the triangle OPC is equilateral. Hence x co-ordinate of P is $3/2$ so that co-ordinates of P are $(3/2, 3\sqrt{3}/2)$
Area includes the boundary

Mathematical behaviours	Marks
• draws the circle with correct centre and correct radius	1+1
• draws lines of appropriate slopes through the origin	1
• shades (or somehow indicates) the required area	1
• derives the co-ordinates of Q	1
• derives the co-ordinates of P	1+1
• makes statement or shows that the boundary is included in region	1

Question 13 (a) **(7 marks)**

Solution	
Initial velocity $\mathbf{v}(0)=140\cos 45^o\,\mathbf{i} + +140\sin 45^o\,\mathbf{j} = 70\sqrt{2}\,(\mathbf{i}+\mathbf{j})$ As $\mathbf{a}(t)=-9.8\mathbf{j}$ then $\mathbf{v}(t)=\int -9.8\mathbf{j}\;dt=-9.8t\,\mathbf{j}+\mathbf{c}$ Applying the initial condition yields $\mathbf{c}=70\sqrt{2}\,(\mathbf{i}+\mathbf{j})$ so $\mathbf{v}(t)=70\sqrt{2}\,\mathbf{i}+(70\sqrt{2}-9.8t)\,\mathbf{j}$ Integrating again then $\mathbf{r}(t)=\int \mathbf{v}(t)dt=70\sqrt{2}t\;\mathbf{i}+(70\sqrt{2}t-4.9t^2)\,\mathbf{j}+\mathbf{d}$ for some constant **d** Since initially **r=0** so **d=0,** and hence $\mathbf{r}(t)=70\sqrt{2}t\;\mathbf{i}+(70\sqrt{2}t-4.9t^2)\,\mathbf{j}$	
Mathematical behaviours	**Marks**
• correctly obtains $\mathbf{v}(0)$	1
• uses $\mathbf{a}(t)=-9.8\mathbf{j}$ to derive $\mathbf{v}(t)$	1
• uses initial conditions to determine the constant vector **c**	1
• deduces that $\mathbf{v}(t)=70\sqrt{2}\,\mathbf{i}+(70\sqrt{2}-9.8t)\,\mathbf{j}$	1
• anti-differentiates $\mathbf{v}(t)$ to give $\mathbf{r}(t)$	1
• uses $\mathbf{r}(0)=\mathbf{0}$ to determine **d**	1
• deduces the final form of $\mathbf{r}(t)$	1

Question 13 (b) **(2 marks)**

Solution	
Maximum height is achieved when the vertical component of velocity vanishes This occurs when $t=70\sqrt{2}/9.8$ At this time height of projectile is $70\sqrt{2}\left(\frac{70\sqrt{2}}{9.8}\right)-4.9\left(\frac{70\sqrt{2}}{9.8}\right)^2=500$ metres	
Mathematical behaviours	**Marks**
• determines time for maximum height	1
• correctly evaluates for maximum height of 500 m	1

Question 13 (c) **(2 marks)**

Solution	
Projectile reaches horizontal again when the **j** component of $\mathbf{r}(t)$ vanishes. The requisite time is $70\sqrt{2}/4.9\approx 20.2$ seconds	
	Marks
• recognition of the need to determine when the vertical displacement is zero	1
• correct calculation of total flight time	1

Question 13 (d) **(2 marks)**

Solution
When $t = 70\sqrt{2}/4.9$ then $\mathbf{v} = 70\sqrt{2}\,(\mathbf{i} - \mathbf{j})$ and speed $= 70\sqrt{2}\sqrt{2} = 140$ m/s

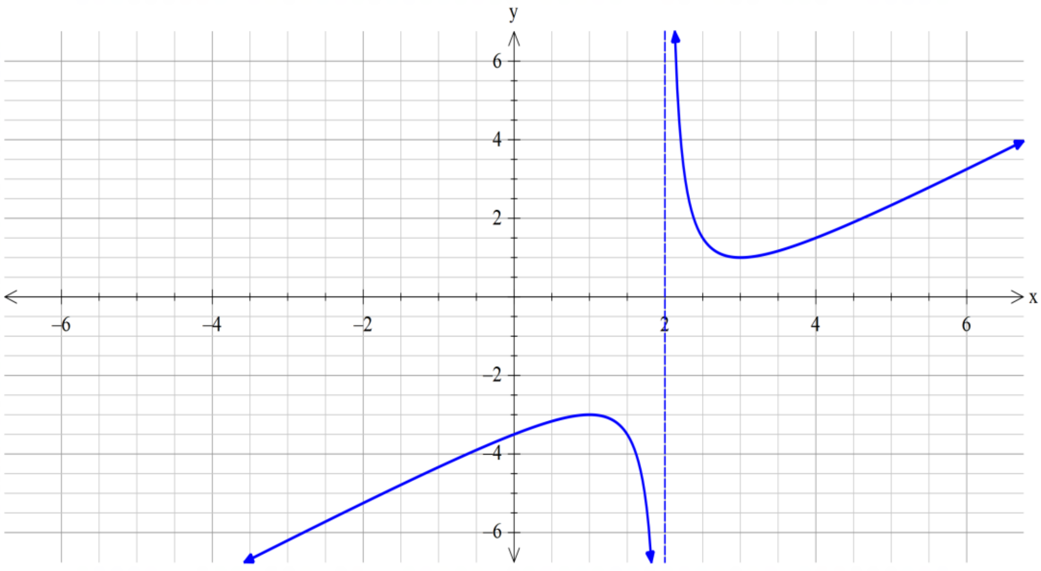
	Marks
• evaluation of the velocity vector when the projectile strikes ground	1
• evaluates the requisite speed	1

Question 14 (a) **(7 marks)**

Solution
Since $f(x)=\dfrac{x^2-5x+7}{x-2}$, $f(x)=0 \Leftrightarrow x^2-5x+7=0$ .But this quadratic has no real roots so $f(x)$ has no real zero. Now $$f'(x)=\frac{(x-2)(2x-5)-(x^2-5x+7)}{(x^2-5x+7)^2}=\frac{x^2-4x+3}{(x^2-5x+7)^2}=\frac{(x-1)(x-3)}{(x^2-5x+7)^2}.$$ Hence $f'(x)=0$ at $x=1$ and $x=3$. Since $f(1)=-3$ and $f(3)=1$, then $(1,-3)$ and $(3,1)$ are critical points. The line $x=-2$ is a vertical asymptote for the graph. As $$f(x)=\frac{x(x-2)-3(x-2)+1}{x-2}=x-3+\frac{1}{x-2}$$ then $f(x)\to x-3$ as $x\to\pm\infty$

Mathematical behaviours	Marks
• shows that there is no zero	1
• differentiates correctly	1
• solves $f'(0)=0$	1
• evaluates $f(1)$ and $f(3)$	1
• obtains vertical asymptote	1
• derives the correct form of the expression for large $\lvert x \rvert$	1
• deduces correct limiting behaviours as $x\to\infty$ and $x\to-\infty$	1+1

Question 14 (b) **(5 marks)**

Solution


Mathematical behaviours	Marks
• shows vertical asymptote correctly	1
• shows critical points correctly	1+1
• indicates where the curve cuts the y-axis	1
• displays oblique asymptote correctly	1

Question 15(a) **(3 marks)**

Solution	
We know that $\exp(i\phi)=\cos\phi+i\sin\phi$ so that $\exp(-i\phi)=\cos(-\phi)+i\sin(-\phi)=\cos\phi-i\sin\phi$ Then $$\exp(i\phi)-\exp(-i\phi)=\cos\phi+i\sin\phi-[\cos\phi-i\sin\phi]=2i\sin\phi$$ so that $$\sin\phi=\frac{1}{2i}\left[\exp(i\phi)-\exp(-i\phi)\right]$$ as required.	
Mathematical behaviours	**Marks**
• writes down appropriate form for $\exp(-i\phi)$	1
• uses appropriate properties to write this exponential in terms of $\cos\phi$ and $\sin\phi$	1
• deduces the requisite result	1

Question 15(b) **(6 marks)**

Solution	
If we denote $$E\equiv\exp(i\phi)\Rightarrow 2i\sin\phi=E-E^{-1}$$ Raising to the appropriate power gives $$32i^5\sin^5\phi=E^5+5E^4(-E^{-1})+10E^3(-E^{-1})^2+10E^2(-E^{-1})^3+5E(-E^{-1})^4+(-E^{-1})^5$$ $$=(E^5-E^{-5})-5(E^3-E^{-3})+10(E-E^{-1})$$ Hence $$32i\sin^5\phi=2i\sin5\phi-10i\sin3\phi+20i\sin\phi$$ so that $$16\sin^5\phi=\sin5\phi-5\sin3\phi+10\sin\phi$$ as required	
Mathematical behaviours	**Marks**
• raises the result of part (a) to the fifth power	1
• expands the binomial correctly (both coefficients and signs)	1+1
• collects powers together in the appropriate way	1
• writes E^n-E^{-n} in terms of $\sin n\phi$	1
• deduces the required answer	1

Question 16(a) **(6 marks)**

Solution
If $z-ki$ is a factor of $P(z)$ then this means that $P(ik)=0$ Hence $k^4+2ik^3-qk^2-98ik+98=0$ …………(*) Equating imaginary parts tells us that $2k^3-98k=0 \Rightarrow 2k(k^2-49)=0$ Now this gives that either $k=0$ or $k=\pm 7$ The real part of (*) gives that $k^4-qk^2+98=0$ Now if $k=0$ this equation cannot hold so this possibility must be ruled out. Then if $k^2=49 \Rightarrow (49)^2-49q+98=0 \Rightarrow 49-q+2=0$ Hence we conclude that $q=51$ and $k=\pm 7$

Mathematical behaviours	Marks
• appreciates that $z-ki$ is a factor implies that $P(ik)=0$	1
• expands the form of $P(ik)$	1
• puts real and imaginary parts of the expression both equal to zero	1
• shows that imaginary part yields three possible values for k	1
• argues that the form of the real part prohibits the possibility $k=0$	1
• hence deduces the appropriate value of q	1

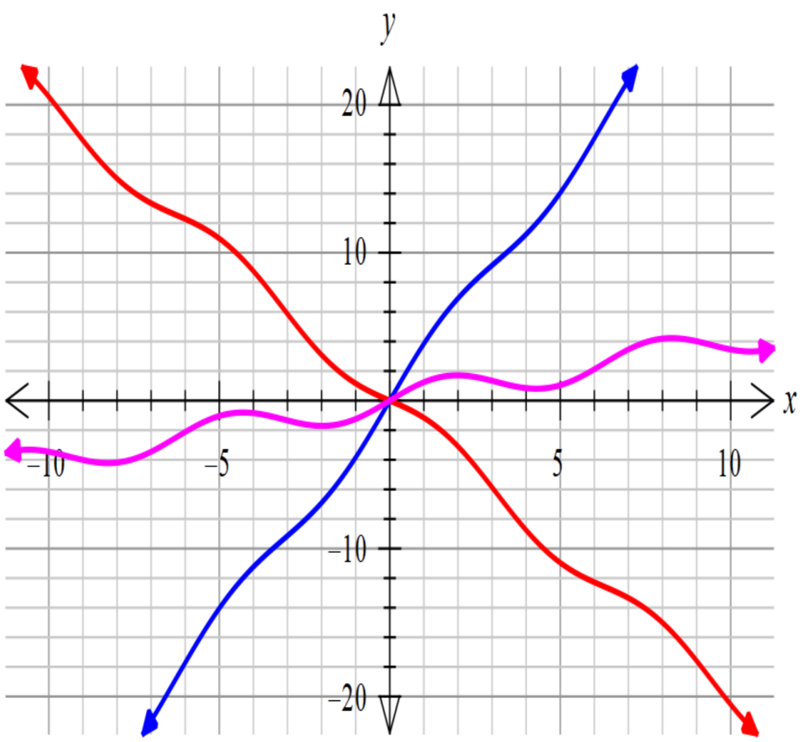
Question 16(b) **(4 marks)**

Solution
Now we deduce that $(z-7i)(z+7i)=(z^2+49)$ is a factor of $P(z)$ By long division or CAS $z^4-2z^3+51z^2-98z+98=(z^2+49)(z^2-2z+2)$ If $z^2-2z+2=0 \Rightarrow z=\frac{1}{2}(2\pm\sqrt{4-8})=1\pm i$ Hence the four roots of $P(z)=0$ are $z=\pm 7i$ and $z=1\pm i$

Mathematical behaviours	Marks
• identifies the quadratic factor of $P(z)$	1
• determines the other quadratic factor	1
• solves for the other two roots of the equation	1
• states all four solutions in explicit form	1

Question 17 (a) **(3 marks)**

Solution



Mathematical behaviours	Marks
• plots the three graphs reasonably accurately (blue curve is $A=3$, purple $A=0.4$ and red $A=-2$)	1+1+1

Question 17(b) **(4 marks)**

Solution

$f'(x) = A + \cos x$
So $A - 1 \leq f'(x) \leq A + 1$.
Now f is $1-1 \leftrightarrow f'$ does no change sign. (*)
So f is $1-1 \leftrightarrow A \geq 1$ or $A \leq -1$

Mathematical behaviours	Marks
• differentiates correctly	1
• notes the range of values of the derivative	1
• identifies criterion for $1-1$ property (*)	1
• answers correctly	1

Question 17(c) **(3 marks)**

Solution	
To evaluate $f^{-1}(5)$ we need to solve $f(x) = -2x + \sin x = 5$ (*) From a calculator $x \cong -2.71$	
Mathematical behaviours	**Marks**
• writes down equation (*) • gives solution to the specified level of accuracy	1 2

Question 18 (a) **(6 marks)**

Solution	
Let A denote the airplane. Given the position and velocity in the question, at time t after 1pm the airplane is located at $(60,160,3.74)+t(-100,20,0.8)=(60-100t,160+20t,3.74+0.8t)$ Similarly, at the same time the helicopter is located at $(-70,108,4.52)+t(-50,40,0.5)=(-70-50t,108+40t,4.52+0.5t)$ The first components of the two position vectors coincide when $60-100t=-70-50t \quad \Rightarrow \quad 130=50t \quad \Rightarrow \quad t=2.6$ When $t=2.6$ the aircraft is located at $(60-260,160+52,3.74+2.08)=(-200,212,5.82)$ The helicopter is located at $(-70-130,108+104,4.52+1.3)=(-200,212,5.82)$	
Mathematical behaviours	**Marks**
• correctly determines the position of the airplane at time t • correctly determines the position of the helicopter at time t • determines when one the components of the two locations are equal • shows that at this time the other two components are also equal • concludes with evidence that a collision is imminent	1 1 2 1 1

Question 18 (b) **(2 marks)**

Solution

Since $t = 2.6$ the collision occurs at 3.36pm and at the location $-200\,\mathbf{i} + 212\,\mathbf{j} + 5.82\,\mathbf{k}$

Mathematical behaviours	Marks
• correctly converts $t = 2.6$ to 3.36 p.m.	1
• states position of the collision point	1

Question 18 (c) **(7 marks)**

Solution

At 2 p.m., $t = 1$ so

$r_A(1) = < 60, 160, 3.74 > +1 < -100, 20, 0.8 > = < -40, 180, 4.54 > km$ and

$r_H(1) = < -70, 108, 4.52 > +1 < -50, 40, 0.5 > = < -120, 148, 5.02 > km$

Making the helicopter to be at rest (by imposing a negative velocity on it) we have:
${}_A v_H = v_A - v_H = < -150, 120, 0.5 > \; - < -50, 40, 0.5 > = < -100, 80, 0 >$
$\overrightarrow{HA} = \overrightarrow{HO} + \overrightarrow{OA} = \; - < -120, 148, 5.02 > + < -40, 180, 4.54 > = < 80, 32, -0.48 >$

$\overrightarrow{HR} = \overrightarrow{HA} + {}_A v_H$ (where *R* is the point at which airplane and helicopter are closest)
$= < 80, 32, -0.48 > + t < -100, 80, 0 >$
$= < 80 - 100t, 32 + 80t, -0.48 >$

Calculate $\overrightarrow{HR} \cdot {}_A v_H = 0$ to determine closest distance between aircraft
i.e. $< 80 - 100t, 32 + 80t, -0.48 > \cdot < -100, 80, 0 > = 0$

Simplifying gives: $5440 = 16400t \rightarrow t = 0.3317$ hours

At $t = 0.3317$, $|\overrightarrow{HR}| = |< 46.83, 58.536, -0.48 >| = 74.96$ km

∴ the shortest distance between the aircraft following the redirection is 74.96 km

Mathematical behaviours	Marks
• determines the position vectors $r_A(1)$ and $r_H(1)$	2
• determines relative velocity vector for ${}_A v_H$	1
• correctly develops equation (i.e. $\overrightarrow{HR} = \overrightarrow{HA} + {}_A v_H$)	1
• uses scalar dot product $\overrightarrow{HR} \cdot {}_A v_H = 0$	1
• evaluates for $t = 0.7508$ hours	1
• determines minimum distance following the redirection	1